since the heat transfer becomes a constant the Pr-dependence does not affect the temperature distribution which is governed by the solution far from the wall. To show that the right of equation (12) approaches a constant for $Pr \to 0$, consider the matched asymptotic solutions of the velocity distribution. In the usual limiting process Pr disappears from (7) and (8), yielding the outer solution which is independent of Pr. For the inner solution, using the fact that $\theta(\eta; Pr) \to \theta(0)$, independent of Pr, the explicit Pr-dependence disappears from equation (7) provided the new variable $Pr^{-1/2}\zeta$ as function of $Pr^{-1/2}\eta$ is employed. Consequently, $Pr^{1/2}\zeta''(0; Pr)$ becomes independent of Pr as $Pr \to 0$, suggesting the expression for dimensionless friction given by equation (12).

The remaining Pr-dependence in dimensionless friction is small, as seen from Fig. 1, and appears to be analogous to that of the heat transfer. The Pr-dependence may of course be further reduced by empirical fit to computer results in a manner similar to that discussed for the heat transfer.

The distributions of dimensionless temperature and velocity of Figs. 2 and 3 show how the present dimensionless variables nearly eliminate variations in wall gradients. The significant changes in velocity distributions with Prandtl number far from the wall appear to have a marginal effect on temperature distributions which prove to be nearly similar for all fluids.

Finally, it is noted from the ratio of equations (11) and (12) that an appropriate parameter group expressing an analogy between heat transfer and friction is, for the present problem,

$$\frac{Nu_x \, Ra_x^{1/2}}{(\tau_w/\rho)(x^2/a\nu)}. \tag{13}$$

This parameter group varies less than 10% over the complete range of Prandtl numbers.

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Mixed convection from a horizontal line source of heat

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INTRODUCTION

THE MECHANISMS of mixed convection from a horizontal line source of heat have come under active investigation relatively recently. The earliest study by Wood [1], followed by that of Wesseling [2], considered mixed convection from weakly buoyant plumes. Afzal [3] presented a complete analysis describing the entire flow regime, ranging from weakly to strongly buoyant plumes. It is well known that in a free convection plume from a line source of heat, the centerline velocity increases continuously as $x^{1/5}$, x being the distance downstream of the source. Thus, even in the presence of a non-zero free-stream velocity, it appears that, at sufficiently large x, buoyancy effects would dominate the transport mechanism. This corresponds to a strongly buoyant plume. Such is the flow studied here, with the buoyancy force and the free-stream flow being in the same direction.

Afzal [3] considered the mixed convection from a line source in terms of two coordinate expansions, a direct coordinate expansion and an inverse, valid for small and large streamwise distance from the source, respectively. The direct coordinate expansion was taken in terms of a variable ξ, ξ being proportional to $x^{1/2}$. The inverse coordinate expansion was constructed in terms of $\xi^{-2/5}$. The solution for the first 11 and the first seven terms, in the direct and inverse coordinate expansions, respectively, are presented. However, these expansions were constructed entirely on the basis of the classical boundary-layer solution. The non-boundary-layer effects, for example, the effect of the flow in the ambient, resulting from the entrainment into the boundary layer, was not considered. Under some circumstances these non-boundary-layer effects also contribute significantly. It then becomes necessary to simultaneously assess both the effects of the presence of external free-stream velocity and of the non-boundary-layer effects. The first such simultaneous and consistent assessment in the analysis of mixed convection flows was that by Carey and Gebhart [4].

The method of matched asymptotic expansions is used here to obtain a solution valid at a large downstream distance from the source. Two perturbation parameters, ε_H and ε_M, characterize the non-boundary-layer and the non-zero free-stream velocity effects, respectively. These effects are considered simultaneously, as perturbations of the associated natural convection plume flow. It is shown that corrections due to higher-order effects enter into the expansion before the fifth term of the inverse coordinate expansion considered by Afzal [3], for the same flow. Results of the numerical computations are presented for $Pr = 0.7$.

ANALYSIS

The mixed convection flow arising from an infinitely long horizontal line source of heat is considered as a two-dimensional steady flow. With usual Boussinesq approximations, neglecting the viscous dissipation and pressure terms in the energy equation, the full two-dimensional governing equations take the form

$$\psi_y \frac{\partial}{\partial x}(\psi_{xx}+\psi_{yy}) - \psi_x \frac{\partial}{\partial x}(\psi_{xx}+\psi_{yy}) - \nu\nabla^4\psi - g\beta\frac{\partial t}{\partial y} = 0 \quad (1)$$

NOMENCLATURE

A_i constants, defined in (19a)–(19d) and (22); $i = 0, 1, 2, 3, 4$
C_p specific heat
D $(g\beta N/4\nu^2)^{-5/12}$
F_i terms in the inner expansion of the stream function ψ; $i = 0, 1, 2, 3, 4$
g acceleration due to gravity
G $4(Gr_x/4)^{1/4}$
G_i terms in the inner expansion of H; $i = 3, 4$
Gr_x $(gx^3/\nu^2)\beta\Delta T$
H $(t-t_\infty)/\Delta T$
H_i terms in the inner expansion of H; $i = 0, 1, 2, 3, 4$
I_0 $\int_0^\infty F_0' H_0 \, d\eta$
J_i terms in the inner expansion of the stream function ψ; $i = 3, 4$
k thermal conductivity
N $(Q_0^4/g\beta\rho^2\mu^2C_p^4I_0^4)^{1/5}/4$
Pr Prandtl number
Q local thermal convected energy in the boundary layer
Q_0 thermal input per unit length of the line source
r radial polar coordinate measured from the heat source
$\bar{R}$ $2U_\infty/\nu(4\,Pr\,I_0 k\nu^2/g\beta Q_0)^{1/3}$
Re_x $U_\infty x/\nu$
t temperature
t_0 centerline temperature
t_∞ ambient temperature
u velocity in the x-direction
u_∞ free-stream velocity
v velocity in the y-direction
x coordinate in the direction opposite to gravity
y coordinate normal to x.

Greek symbols
β coefficient thermal expansion
ΔT $Nx^{-3/5}$
ε_H $(Gr_x/4)^{-1/4}$
ε_M $4\,Re_x/G^2$
η $yG/4x$
θ $t-t_\infty$, in the inner expansion
$\bar{\theta}$ $t-t_\infty$, in the outer expansion
μ coefficient of viscosity
ν kinematic viscosity
ρ density
ϕ $\tan^{-1}(y/x)$
ψ streamfunction
$\bar{\psi}$ streamfunction in the outer expansion
$\bar{\psi}_i$ terms in the expansion for $\bar{\psi}$; $i = 1, 2, \ldots$.

and

$$\psi_y \frac{\partial t}{\partial x} - \psi_x \frac{\partial t}{\partial y} = \frac{\nu}{Pr}(t_{xx}+t_{yy}) \quad (2)$$

where the streamfunction ψ has been defined so that

$$U = \psi_y \quad \text{and} \quad V = -\psi_x.$$

Boundary conditions are:

$$y = 0, \quad \psi = \psi_{yy} = t_y = 0; \quad \text{for all } x \quad (3a)$$

$$y \to \infty, \quad \psi_y \to U_\infty, \quad t \to t_\infty; \quad \text{for all } x. \quad (3b)$$

Also, for $x > 0$, the convected energy is

$$Q(x) = \int_{-\infty}^{\infty} \rho C_p \psi_y (t-t_\infty)\,dy = Q_0 = \text{constant} \quad (4)$$

where Q_0 is the thermal input per unit length of the line source, ρ is the density and C_p is the specific heat.

The inner and outer expansions are then taken as:

Inner

$$\psi = 4\nu\sqrt[4]{Gr_x/4}[F_0(\eta)+\varepsilon_M F_1(\eta)+\varepsilon_M^2 F_2(\eta)+\varepsilon_M^3 F_3(\eta) + \varepsilon_H J_3(\eta)+\varepsilon_M^4 F_4(\eta)+\varepsilon_M\varepsilon_H J_4(\eta)+\cdots] \quad (5)$$

$$\theta \equiv t-t_\infty = \Delta T[H_0(\eta)+\varepsilon_M H_1(\eta)+\varepsilon_M^2 H_2(\eta) + \varepsilon_M^3 H_3(\eta)+\varepsilon_H G_3(\eta)+\varepsilon_M^4 H_4(\eta)+\varepsilon_M\varepsilon_H G_4(\eta)+\cdots] \quad (6)$$

outer

$$\bar{\psi} = \bar{\psi}_0+\bar{\psi}_1+\bar{\psi}_2+\bar{\psi}_3+\cdots \quad (7)$$

$$\bar{\theta} = t-t_\infty \equiv 0 \quad (8)$$

where $\Delta T = Nx^{-3/5}$ represents the temperature difference between the local centerline of the plume and the ambient, resulting from the zeroth-order boundary-layer solution, given by F_0 and H_0 for velocity and temperature fields, respectively. The following quantities are defined:

$$Gr_x = \frac{g\beta x^3\Delta T}{\nu^2} = \left(\frac{g\beta N}{\nu^2}\right)x^{12/5}; \quad \eta = \frac{y}{x}\sqrt[4]{\frac{Gr_x}{4}} = \frac{y}{4x}G$$

$$N = (Q_0^4/4^5 g\beta\rho^2\mu^2 C_p^4 I_0^4)^{1/5}; \quad I_0 = \int_0^\infty F_0' H_0\,d\eta$$

and μ is the dynamic viscosity.

Here, ε_M is taken to be $Re_x/(Gr_x/4)^{1/2}$ so that F_0 and H_0, are the solutions of the natural convection boundary-layer flow shed from a line source in a quiescent ambient medium. Following Yang and Jerger [5], ε_H is chosen as $(Gr_x/4)^{-1/4}$. Then ε_M and ε_H are related as:

$$\varepsilon_M = \bar{R}\varepsilon_H^{1/3}$$

where

$$\bar{R} = 2U_\infty/\nu(4Pr\,I_0 k\nu^2/g\beta Q_0)^{1/3}.$$

The parameter $\bar{R}$ is important in that if $\bar{R} \gg 1$ then the interaction terms corresponding to $\varepsilon_M\varepsilon_H$ would be negligible and the solution could be obtained by appropriately superposing the solutions of Afzal [3] and Riley [6]. However, for ordinary ranges of Q_0 and U_∞ encountered, of $Q_0 \sim 50\ \mathrm{W\,m^{-1}}$ and $U_\infty \sim 1$–$10\ \mathrm{cm\,s^{-1}}$, $\bar{R}$ turns out to be a constant of the order of unity.

The expansions (5) and (6) are now substituted into the governing equations (1) and (2). Perturbation equations are then obtained by collecting like powers of ε_M, ε_H and $\varepsilon_M\varepsilon_H$.

Employing usual asymptotic matching techniques, boundary conditions are obtained for each level of expansion in the inner and outer regions. After matching, it is found that the terms of inner expansion must satisfy the following equations.

$$F_0''' + 12/5F_0F_0'' - 4/5F_0'^2 + H_0 = 0 \quad (9a)$$

$$H_0'' + 12/5\,Pr\,(F_0H_0)' = 0 \quad (9b)$$

$$F_0(0) = F_0''(0) = F_0'(\infty) = H_0(0)-1 = H_0'(0) = 0. \quad (9c)$$

In the manner of Afzal [3], the governing equations up to $O(\varepsilon_M^4)$ can be expressed as:

$$L_{1n}(F_n, H_n) = R_{1n}, \quad n = 1, 2, 3, 4 \tag{10}$$

$$L_{2n}(F_n, H_n) = R_{2n}, \quad n = 1, 2, 3, 4 \tag{11}$$

where the operators, L_{1n} and L_{2n} are defined as

$$L_{1n}(F_n, H_n) = F_n''' + 4\left(\frac{3}{5}F_0F_n'' + \frac{n-2}{5}F_0'F_n' + \frac{3-n}{5}F_0''F_n + H_n\right) \tag{12}$$

$$L_{2n}(F_n, H_n) = Pr^{-1}H_n'' + 4\left(\frac{3}{5}F_0H_n' + \frac{3+n}{5}F_0'H_n + \frac{3-n}{5}F_nH_0' + \frac{3}{5}F_n'H_0\right) \tag{13}$$

$$5R_{1n} = -4\sum_{r=1}^{n-1}[(3-r)F_rF_{n-r}'' + (r-1)F_r'F_{n-r}'] \tag{14}$$

$$5R_{2n} = -4\sum_{r=1}^{n-1}[(3+r)H_rF_{n-r}' + (3-r)F_rH_{n-r}']. \tag{15}$$

The boundary conditions being

$$F_n(0) = F_n''(0) = H_n'(0) = H_n(\infty) = F_n'(\infty) - \frac{1}{4}\delta_{1n} = 0 \tag{16}$$

where δ_{1n} is the Kronecker delta.
The governing equations at $O(\varepsilon_H)$ and $O(\varepsilon_M\varepsilon_H)$ are given by:

$$J_3''' + \frac{4}{5}(3F_0J_3'' + F_0'J_3') + G_3 = 0 \tag{17a}$$

$$G_3'' + \frac{4}{5}Pr(3F_0G_3' + 6G_3'F_0' + 3H_0J_3') = 0 \tag{17b}$$

$$J_3(0) = J_3''(0) = J_3'(\infty) - \frac{3}{5}A_0\cot(2\pi/5) = G_3'(0) = G_3(\infty) = 0 \tag{17c}$$

and

$$J_4''' + \frac{4}{5}(-J_4F_0'' + 2F_0'J_4' + 3F_0J_4'' + 2F_1J_3'' + 2J_3'F_1') + G_4 = \frac{6}{25}A_0\cot(2\pi/5) \tag{18a}$$

$$G_4'' + \frac{4}{5}Pr(-J_4H_0' + 2F_1G_3' + 3F_0G_4' + 3H_0J_4' + 7G_4F_0' + 6G_3F_1' + 4H_1J_3') = 0 \tag{18b}$$

$$J_4(0) = J_4''(0) = J_4'(\infty) + \frac{2}{5}A_1\cot(2\pi/5) = G_4'(0) = G_4(\infty) = 0 \tag{18c}$$

where

$$F_0(\eta) \sim A_0 \quad \text{as} \quad \eta \to \infty \tag{19a}$$

$$F_1(\eta) \sim (1/4)\eta + A_1 \quad \text{as} \quad \eta \to \infty \tag{19b}$$

$$J_3(\eta) \sim (3/5)A_0\cot(2\pi/5)\eta + A_3 \quad \text{as} \quad \eta \to \infty \tag{19c}$$

$$J_4(\eta) \sim -2/5A_1\cot(2\pi/5)\eta + A_4 \quad \text{as} \quad \eta \to \infty. \tag{19d}$$

In the outer inviscid region, $\bar{\psi}_0$ results merely from the presence of a free stream and is thus given by

$$\bar{\psi}_0 = U_\infty y = U_\infty r\sin\phi \tag{20}$$

where r is the radial distance from the heat source and ϕ is the angular displacement from the positive x-axis.

From matching considerations, the remaining terms of the outer expansion must satisfy,

$$\nabla^2\bar{\psi}_i = 0; \quad i = 1, 2, 3 \tag{21a}$$

$$\bar{\psi}_i|_{\phi=0} = 4\nu A_{i-1}\bar{R}^{i-1}(r/D)^{3/5}, \quad \bar{\psi}_i|_{\phi=\pi} = 0; \quad i = 1, 2, 3 \tag{21b}$$

where

$$D = (g\beta N/4\nu^2)^{-5/12}, \quad A_2 = F_2(\infty). \tag{22}$$

Solving (21a) and (21b), we obtain,

$$\bar{\psi}_i = 4\nu A_{i-1}(r/D)^{(4-i)/5}\sin[(4-i)(\pi-\phi)/5]/\sin[(4-i)\pi/5]; \quad i = 1, 2, 3. \tag{23}$$

It has been pointed out by Afzal [3] that an indeterminancy arises in the inner expansions at $O(\varepsilon_M^5)$ owing to the presence of eigensolutions. However, having restricted ourselves to terms of $O(\varepsilon_M^4)$ in expansions (5) and (6), such an indeterminacy will not be encountered. Thus, the assumed form of the solution in (5) and (6) is appropriate to $O(\varepsilon_M^4)$. The total convected energy $Q(x)$ defined in (4) now becomes, in view of (5) and (6)

$$Q(x) = 4Pr(N^5k^4g\beta/\nu^2)^{1/4}\int_{-\infty}^{\infty}\left\{H_0F_0' + \sum_{n=1}^{4}\varepsilon_M^n\left[\sum_{r=0}^{n}F_r'H_{n-r}\right] + \varepsilon_H(H_0J_3' + G_3F_0') + \varepsilon_M\varepsilon_H(H_0J_4' + F_1'G_3 + F_0'G_4)\right\}d\eta. \tag{24}$$

However, using the energy equation of each level of expansion and the corresponding boundary condition it can be shown that each of the integrals on the RHS of (24) vanishes identically, except for

$$\int_{-\infty}^{\infty} F_0'H_0\,d\eta.$$

Thus, the requirement in (4) that $Q(x)$ be constant is still satisfied by the expansion.

RESULTS AND DISCUSSION

The inner region equations (9a)–(18c) were solved for $Pr = 0.7$ using a modified predictor–corrector method supplemented by a shooting algorithm. Values of $F_i'(0)$, $H_i(0)$, $i = 1, 2, 3, 4$; and $J_3'(0)$, $J_4'(0)$, $G_3(0)$ and $G_4(0)$ were guessed and subsequently corrected so as to satisfy the far field boundary conditions. A fixed step size of $\Delta\eta = 0.05$ was employed and 'infinity' was taken to be at $\eta = 15$.

Values of $F_i'(0)$, $H_i(0)$, A_i, $i = 1, 2, 3, 4$; and $J_3'(0)$, $J_4'(0)$, $G_3(0)$ and $G_4(0)$ are listed in Table 1. From this table it can be seen

Table 1. Computed constants for mixed convection flow from a line source plume, for $Pr = 0.7$

	i = 0	1	2	3	4		
$F_i'(0)$	0.6618	0.0150	0.0172	0.0043	−0.0003	$J_3'(0) = 0.0420$	$J_4'(0) = 0.0102$
$H_i(0)$	1.0	−0.1550	−0.0018	0.0093	0.0018	$G_3(0) = -0.3002$	$G_4(0) = -0.0339$
A_i	0.9313	−0.4391	0.0905	−0.0952	−0.0492		

that

$$U|_{y=0} = (4\nu/x)(Gr_x/4)^{1/4}[0.6618 + \varepsilon_M(0.015) + \varepsilon_M^2(0.0172) + \varepsilon_M^3(0.0043) + \varepsilon_H(0.042) + \varepsilon_M^4(-0.0003) + \varepsilon_M\varepsilon_H(0.0102)] \quad (25)$$

$$t_0 - t_\infty = Nx^{-3/5}[1 + \varepsilon_M(-0.155) + \varepsilon_M^2(-0.0018) + \varepsilon_M^3(0.0093) + \varepsilon_H(-0.3002) + \varepsilon_M^4(0.0018) + \varepsilon_M\varepsilon_H(-0.0339)]. \quad (26)$$

Also at infinity, we have

$$U|_{y\to\infty} = (4\nu/x)(Gr_x/4)^{1/2} \times [0.25\varepsilon_M + 0.1815\varepsilon_H + 0.057\varepsilon_M\varepsilon_H] \quad (27)$$

For $\bar{R} = 1$, the first five terms of (25) and (26) indicate that the usefulness of the series is limited to about $\varepsilon_M \leqslant 0.4$. In some instances (see ref. [7]) it is possible to extend the usefulness of such slowly convergent asymptotic series by means of Shanks' non-linear transformation [8]. But this transformation is somewhat *ad hoc* in nature and the result so obtained need not necessarily yield an improvement. This question has to be decided by a comparison with the numerical solution to the complete Navier–Stokes equations. This, however, is beyond the scope of the present work.

When $\varepsilon_M = 0.4$ and $\varepsilon_H = 0.004$ the presence of ε_H term increases the velocity by nearly 12% at the edge and by 0.4% at the centerline of the plume. On the other hand the $\varepsilon_M\varepsilon_H$ term affects the velocity at the edge and at the centerline of the plume by 1.3% and 0.04%, respectively. On the centerline temperature, ε_H effects it by only 8% and $\varepsilon_M\varepsilon_H$ term by 0.1%. Thus clearly the effects of $\varepsilon_M\varepsilon_H$ term are negligible in comparison with those of ε_H term. In conclusion we can therefore say that the solution can be conveniently obtained by appropriately superposing the results of [3] and [6].

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